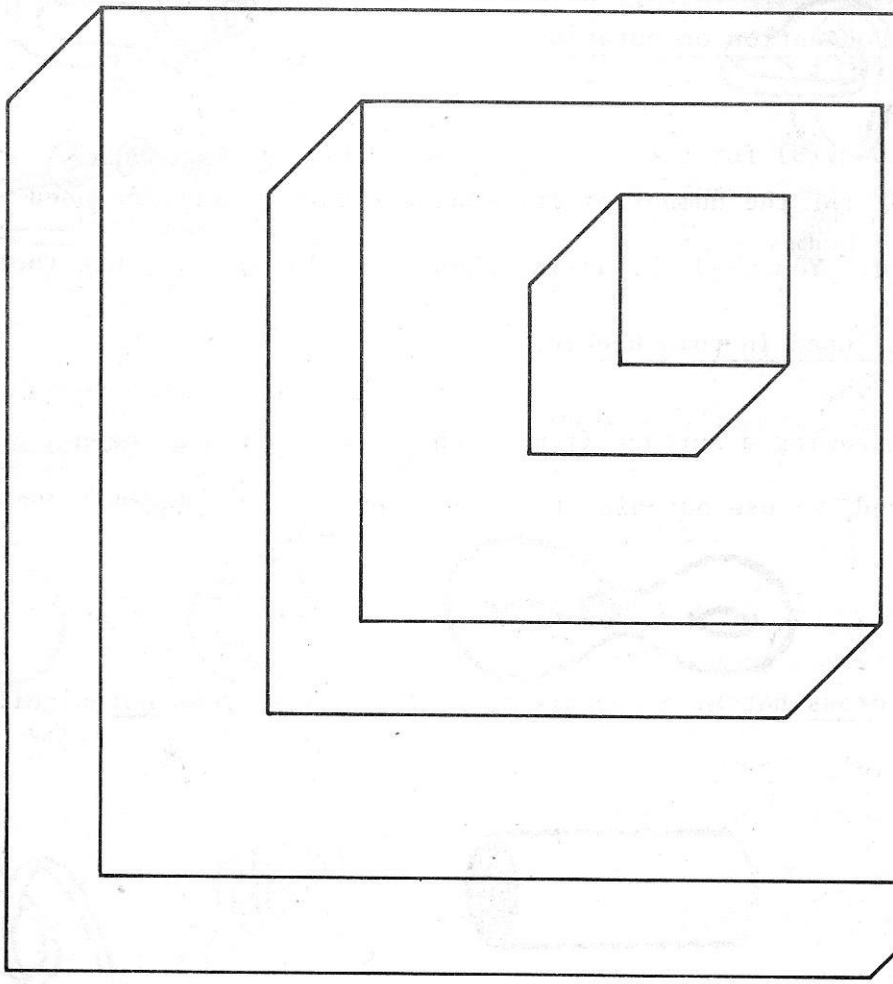
M335 GT EBI

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

STUDIES IN PURE MATHEMATICS

GEOMETRIC TOPOLOGY

EXERCISE BOOKLET I



Copyright © 1983 The Open University

SUP 09764 6

1.3

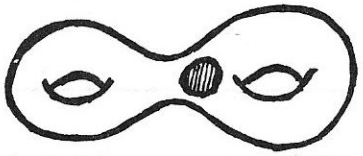
This booklet contains a number of exercises linked to the material of Units 1-4. Some of a routine nature give you practice in basic techniques, and some are more searching. The latter are marked with asterisks. You may wish to treat these as worked examples. We have deliberately provided more exercises than you can possibly attempt as part of your week's work on a unit. If you feel happy with a particular technique then you can skip some of the exercises on it. There should also be ample questions left to act as a stimulus for revision at the end of the course. Full solutions appear at the end of the booklet, and these should give you an idea of how to set out your answers to assignment questions.

Important convention on notation

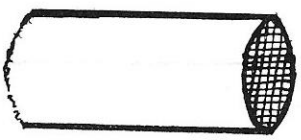
The notation $\chi(S)$ for the Euler characteristic of a surface S, and $\beta(S)$ for the number of its boundary curves, will be used as standard. You need not define these symbols when you use them.

Conventions used in this booklet

(i) When drawing a surface from which an open disc has been removed, we use hatching to denote the "hole", thus



When cross hatching appears (cf. Unit 4) it does not signify a "hole".

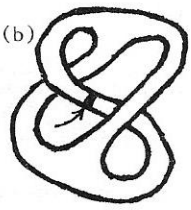


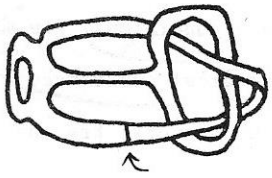
(ii) An n-holed torus is to mean an n-fold torus (i.e. sphere with n handles), as distinct from a torus with n "holes".

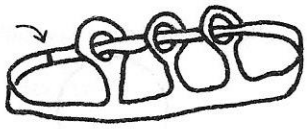
Section 1.1

1. Sketch simpler versions of the following surfaces, which are to be thought of as made from very thin paper. (You may find it helpful to first cut along the indicated line.)

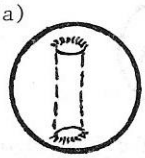
(a)

(b) 

(c) 

(d) 

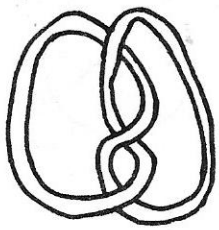
2. Sketch simpler versions of the following surfaces, and give brief indications of how you arrived at them.

(a) 

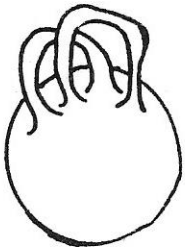
(sphere with a tube through it)

(b)

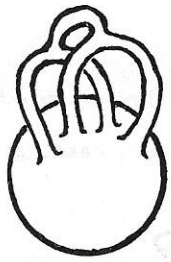
(sphere with a Y-shaped tube inside it)

(c) 

(tubular)

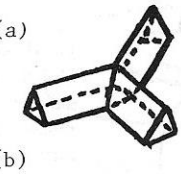
(d) 

(sphere with two handles)

(e) 

(sphere with two handles joined by a handle)

3. Sketch the boundary (if any) of each of the following surfaces.

(a) 

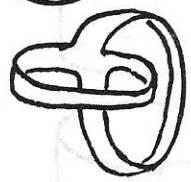
(Y shape — triangular section tube, open at ends)

(b)

(torus)

(c)

(sphere with two open discs removed)

(d) 

(two cylinders stuck together)

4. Write down β, the number of boundary curves, for each of 3(a) - 3(d) above.

Section 1.2

There are no exercises on this section.

Section 1.3

1. Which of the following are

(i) path-connected?

(ii) closed?

(iii) bounded?

(a) $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=1,\ z \geq 0\}$

(b) $\{(x,y) \in \mathbb{R}^2 : 0<r<x^2+y^2<R\}$

(c) $\{(x,y) \in \mathbb{R}^2 : 0<r<x^2+y^2 \leq R\}$

(d) $\{(x,y) \in \mathbb{R}^2 : 0<r \leq x^2+y^2 \leq R\}$

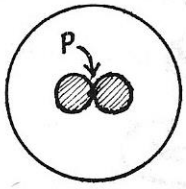
(e) $\mathbb{R}^2$

Section 1.4

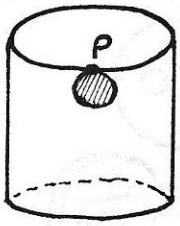
1. None of the following objects are surfaces. In each case, give a brief reason why not.

(a)

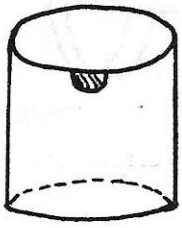
(Finite portion of a double cone with two boundary curves.)

(b) 

(A disc from which two open discs have been removed in such a way that P, which remains, is in the closure of both discs.)

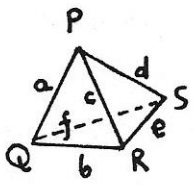
(c) 

(A cylinder with an open disc removed in such a way that the boundary point P is in the closure of the disc.)

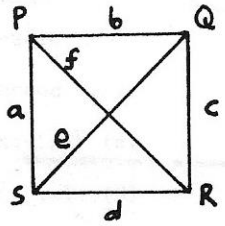
(d) 

(A cylinder with an open disc removed in such a way that a portion of the cylinder's boundary is in the closure of the disc.)

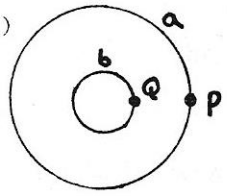
2. Which of the following are valid subdivisions of a surface? (All vertices are marked by capital letters, all edges by lower case letters.) When the surface has not been subdivided, explain briefly why not.

(a) 

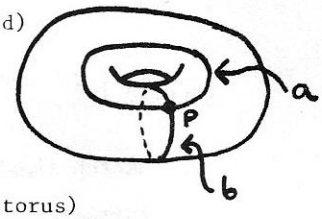
(tetrahedron)

(b) 

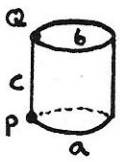
(disc)

(c) 

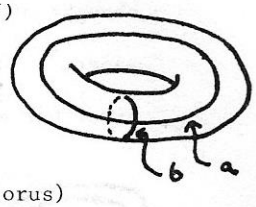
(annulus)

(d) 

(torus)

(e) 

(cylinder)

(f) 

(torus)

3. This exercise is concerned with producing subdivisions of a surface. Conditions 1 and 2 of Frame 14 are matters of book-keeping; condition 4 and to a lesser extent condition 3 are matters of topological substance. There are no unique answers; try and produce simple ones.

(a) Mark a subdivision on a disc.

(b) Mark a subdivision on a disc with a hole;

and then on a disc with two holes.

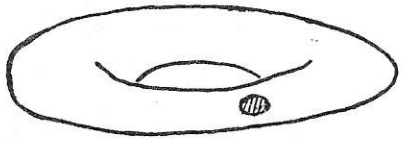
(c) Describe a subdivision on a disc with n holes.

(d) Mark a subdivision on the torus.

(e) Mark a subdivision on the double (two-holed) torus.

(f) Describe a subdivision on the n-holed torus.

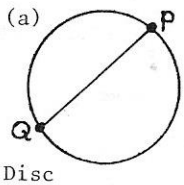
(g) Mark a subdivision on the torus with an open disc removed.



4. This exercise is designed for practice in the art of computing Euler characteristics of surfaces with subdivisions. The subdivisions we have chosen are not necessarily the best possible in any sense and you might like to compare them with your answers to Exercise 3 above. We have not labelled the edges.

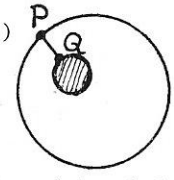
Compute the number of vertices (N_0), edges (N_1) and faces (N_2), and hence the Euler characteristic, of each of the following surfaces.

(a)



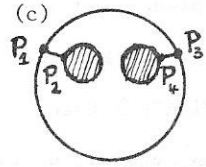
Disc

(b)



Disc with a hole

(c)

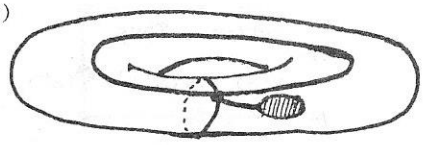


Disc with two holes

(d) Disc with n holes

(e) Torus with the subdivision of 2(d) above

(f)

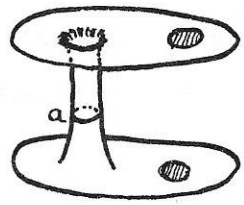


torus with a hole

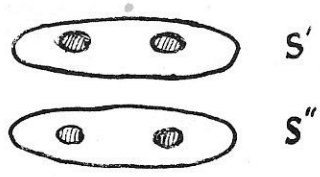
(g) n-holed torus

5. When computing Euler characteristics it is often helpful to use the technique of Section 1.1 to redraw the surface in a more convenient way. Compute the Euler characteristics of the surfaces in Exercises 1 and 2 of Section 1.1 of this booklet.

Sometimes it is helpful to disconnect a surface. For example, to compute the Euler characteristic of this surface



it is perhaps easiest to cut it into two pieces along the edge obtaining



(after squashing the tubes).

Now each piece is a disc with 2 holes, with Euler characteristic -1. To find the Euler characteristic of the original surface, observe that in each of the pieces edge a contributes nothing, because it is one edge with one vertex. Thus

$\chi(S) = \chi(S') + \chi(S'')$

$= -2.$

6. Check directly (by marking a subdivision on S) that $\chi(S) = -2$.

7. The other topological invariant introduced in Unit 1 is β, the number of boundary curves. This can be conveniently counted by using coloured pens.

Find β for each of the surfaces in Exercises 1 and 2 of Section 1.1 above and Exercise 4 of this section.

Section 2.1

1. In this exercise we ask you to investigate the constraints imposed by the Euler characteristic on finding regular subdivisions of a surface (without boundary) with $\chi = -6$.

   (a) Show that if each face has k edges and each vertex has order j, then

   $$\frac{1}{k} + \frac{1}{j} < \frac{1}{2}.$$

   (b) Deduce that for a regular triangulation

   $$7 \leq j \leq 42.$$

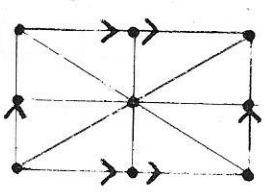
   (c) Prove that if $j = 8$ there is exactly one solution for V, E, F and find their values.

2. Investigate the constraints on finding regular triangulations of a surface (without boundary) of Euler characteristic -2.

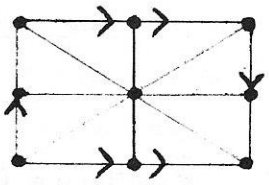
3. Which of the following subdivisions are triangulations? In each case which fails to be a triangulation, state what the failure is and indicate how the subdivision can be improved to become a triangulation.

   (a) The tetrahedral subdivision of the sphere.

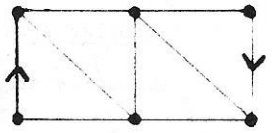
   (b) This subdivision of a torus:



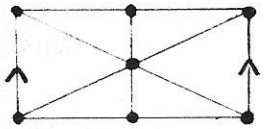
   (c) This subdivision of a Klein bottle:



   (d) This subdivision of a Mobius band:



   (e) This subdivision of a cylinder:



Section 2.2

There are no exercises on this section. However, we recommend that when you feel like some light relief you draw n parallel lines on a strip of paper, give it k half-twists and join the ends. Then cut it along the (how many?) loops so formed. Try to decide the characteristic features of the surfaces you obtain.

Section 2.3

1. For a regular subdivision of the Klein bottle (which has $\chi = 0$), the number of edges on each face (k) and the order of each vertex (j) satisfy

   $$\frac{1}{k} + \frac{1}{j} = \frac{1}{2}$$

   as for a torus.

The solutions are

$j=3,\ k=6;\quad j=k=4;\quad j=6,\ k=3.$

In the first case we have

$6F = 2E,\ 3V = 2E$

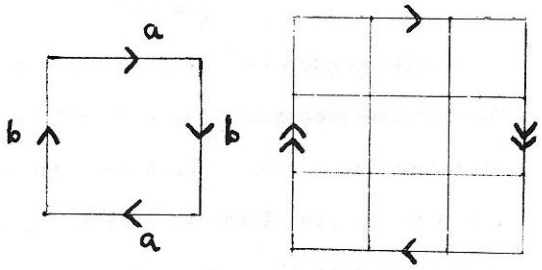
so for each integer N this formula allows a subdivision into N hexagons, having 3N edges and 2N vertices.

(a) Find similarly what the possibilities are in the other two cases.

(b) Draw sketches to show how these might all be obtained.

[Hint: Look at SAQ14 of Unit 2.]

2. (a) Consider the square with opposite edges identified to give a representation of the projective plane.



Subdivide this (as shown above) into 9 small squares, keeping the same identifications. Why does this fail to give a regular subdivision of the projective plane into 9 squares?

(b) Now reverse one of the arrows in each of the above figures to give a representation of the Klein bottle. Do we have a regular subdivision of the Klein bottle into 9 squares?

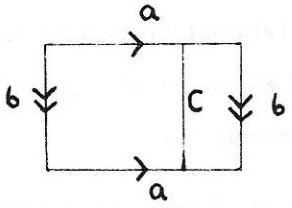
Section 2.4

Orientation-reversing curves

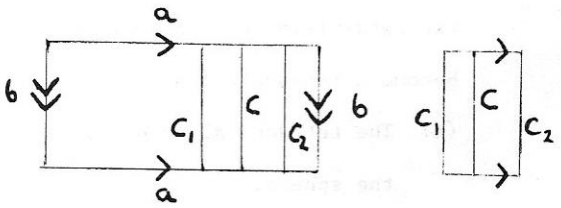
There is a simple technique for telling whether a curve on a surface is orientation-reversing or not. It reverses orientation if, and only if, its neighbourhood is a Möbius band; it preserves orientation if, and only if, its neighbourhood is a cylinder. We shall always assume that the curve under consideration does not meet the boundary, so we analyse neighbourhoods by looking at their boundaries. The Möbius band has a single boundary curve, whilst the cylinder has two.

Example

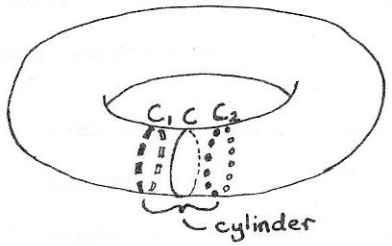
Consider the circle C in a torus:



A neighbourhood of it in the torus is found by drawing a thin strip surrounding C and considering the edge identifications that arise:



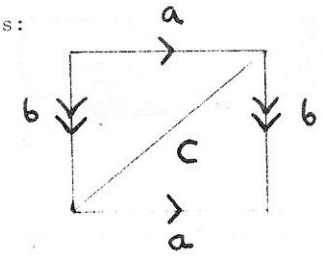
In this case the strip is a cylinder, which it is also possible to see on the torus itself:



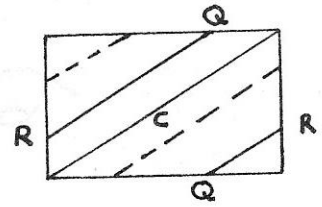
The neighbourhood has two boundary components, C_1 and C_2, and for this reason also it is a cylinder. Thus C does not reverse orientation.

Example

Draw a neighbourhood of this circle on a torus:



Solution

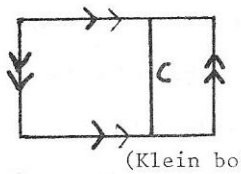


We have two boundary curves(remember the identifications). The neighbourhood is a cylinder, and C does not reverse orientation.

As usual, when counting boundary curves you may find coloured pens useful.

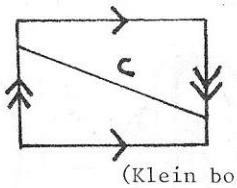
1. Find neighbourhoods of the circles C in each of the following examples; draw them and state whether they are cylinders or Möbius bands.

(a)



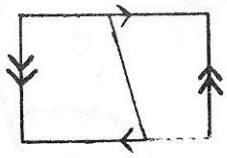
(Klein bottle)

(b)



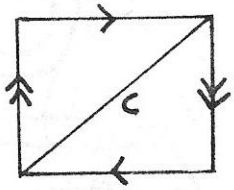
(Klein bottle)

(c)



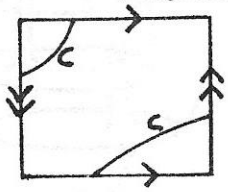
(Projective plane

(d)



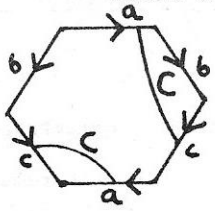
(Projective plane)

(e)



(Klein bottle)

(f)



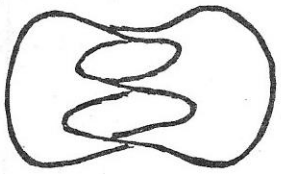
(this is also in fact, a Klein bottle)

When a surface is given pictorially, there are essentially two ways of deciding whether or not it is orientable. One is to sketch it in a simpler form using cutting and glueing operations. The other is to try and draw a Möbius band on it.

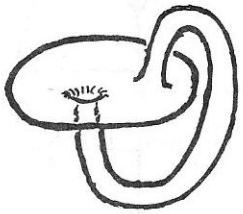
2. For each of the following surfaces, decide whether or not it is orientable. If it is not orientable, sketch a Möbius band on it.

(a)

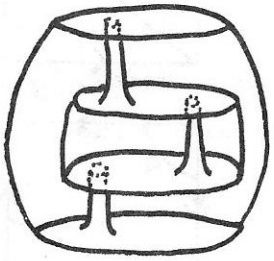
(b)



(c)



(d)



3. Find examples of distinct surfaces which have only one boundary curve.

4. Find counter-examples to the claim that the only surface with two boundary curves is a cylinder. Can you find any non-orientable examples?

Miscellaneous Questions on Units 1 and 2

1. (a)* Show that if a surface is cut along an orientation-reversing curve then it remains path-connected.

   (b) Give examples to show that cutting a surface along an orientation-preserving curve may or may not leave it path-connected.

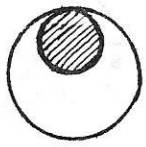
2. (a) Cut a Möbius band along its centre circle. What surface do you obtain?

   (b) Prove your answer to (a).

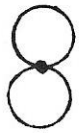
Section 3.1

1. None of the following are surfaces. In each case state which clause(s) in the definition of surface are not satisfied. Which are topological polygons?

(a)

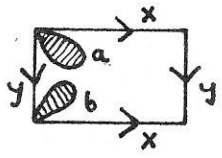


(b)

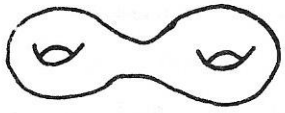


(c)

(d)



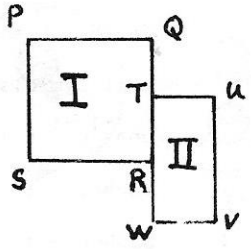
2. Show how the double torus is the continuous image of a decagon



by marking a suitable subdivision on the double torus and labelling the edges. [Hint: Think first of a topological square on a single torus.]

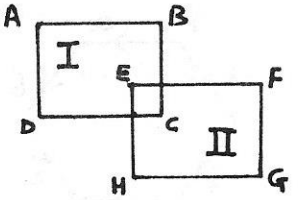
3. Explain why each of the following arrangement of polygons fails to define a surface.

(a)



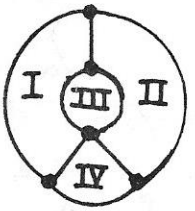
(Two rectangles, edges PQ, QR, RS, SP and TU, UV, VW, WT)

(b)



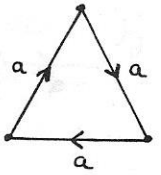
(Two rectangles, vertices A, B, C, D and E, F, G, H)

(c)



(4 faces, 7 edges, 5 vertices)

(d)



(This is known as the "Dunce Hat")

4. Which of the following are edge equations of a surface? What is wrong with those that are not?

(a) $aa = 1$

(b) $abac$

(c) $abacabad = 1$

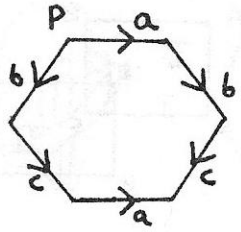
(d) $aba^{-1}baca^{-1}d = 1$

(e) $abcc^{-1}b^{-1}a^{-1} = 1$

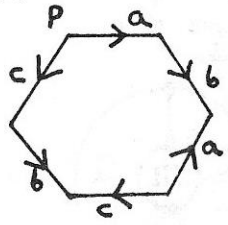
(f) $aa^{-1} = 1$

5. Write edge equations for each of the following surfaces by starting at the vertex P and working clockwise. Do not simplify the equations.

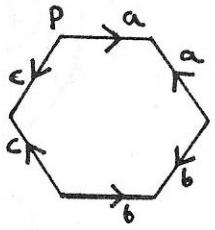
(a)



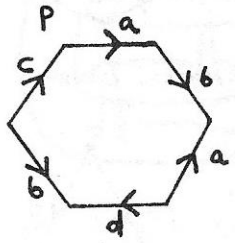
(b)



(c)

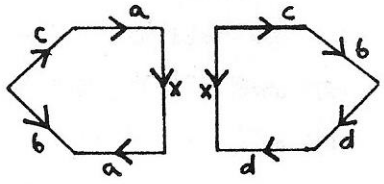


(d)

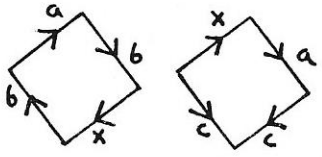


6. In this exercise, various surfaces are presented in the form of polygons with identified edges. Obtain edge equations for each polygon separately, and also for the polygons obtained in each case by identifying the edges labelled x. (Answers are NOT unique.)

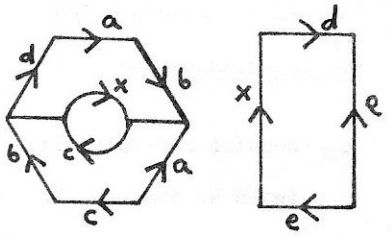
(a)



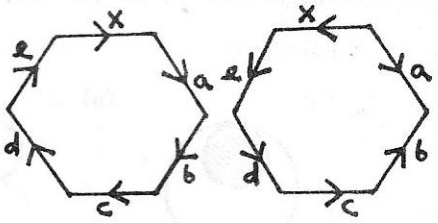
(b)



(c)



(d)



7. Show that the edge equation

$$abacb^{-1}c^{-1} = 1$$

is equivalent to each of the following.

(a) $acb^{-1}c^{-1}ab = 1$

(b) $aba = cbc^{-1}$

(c) $xyxz^{-1}y^{-1}z = 1$

(d) $aby = 1$, $y = acb^{-1}c^{-1}$

Section 3.2

These exercises provide practice in using the rules governing combinatorial equivalence (Blackett pages 63, 64). Remember that if $c = ab$, then $b = a^{-1}c$ (not ca^{-1}) - the order is crucial.

In Exercises 1-7, show that the left-hand edge equation is combinatorially equivalent to the right-hand one.

1.	$abcba = 1$	$aabbc^{-1} = 1$
2.	$Kabcbah = 1$	$Kaabbc\ h = 1$
	(Here K is any block of symbols.)	
3.	$abcb = 1$	$bba^{-1}c^{-1} = 1$
4.*	$abcacb = 1$	$aabbcc = 1$
5.*	$acbca^{-1}b^{-1} = 1$	$aabbcc = 1$

Thus far we have dealt mainly with edge equations of the form

$$AxBxC = 1$$

and brought the edges x together (possibly after re-labelling). We now rearrange equations such as

$$AxByCx^{-1}Dy^{-1}E = 1$$

so that the final equation has

$$xyx^{-1}y^{-1}$$

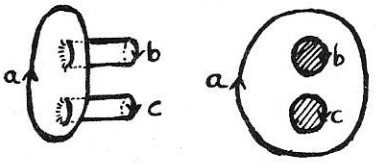
in it somewhere.

6. $axcyx^{-1}y^{-1}F = 1$ $\quad$ $axyx^{-1}y^{-1}cF = 1$

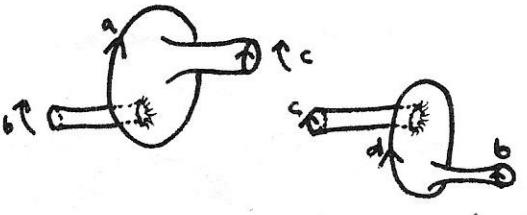
(Here F is any block of symbols.)

7. $axcydx^{-1}y^{-1}F = 1$ $\quad$ $axyx^{-1}y^{-1}dcF = 1.$

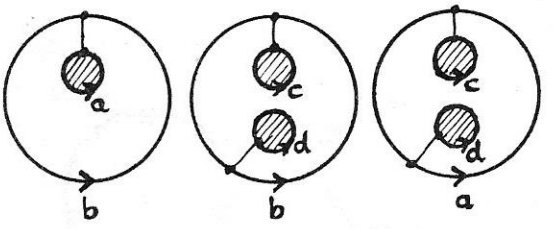
8. The double cylinder (or double annulus)



is homeomorphic to a disc with two holes. What surface is obtained by identifying the pairs of edges marked b and c in the following pair of double annuli?



9. Make a surface out of these pieces.



[Hint: First push out the edges of the holes to form tubes, then try to join up.]

10. Each of the following edge equations defines a surface. In each case state whether or not the surface is orientable, and also list the edges which lie on the boundary of the surface.

(a) $abacbd = 1$

(b) $ababcdc^{-1}d^{-1} = 1$

(c) $aab^{-1}c^{-1}bd = 1$

(d) $abcbcda^{-1}d^{-1} = 1$

(e) $aba^{-1}b^{-1}cded^{-1}e^{-1}f = 1$

(f) $aba^{-1}b^{-1}cdc^{-1}efe^{-1}f^{-1}ghg^{-1} = 1$

Section 3.3

1. The double of a surface M with boundary ∂ ($= \partial M \neq \emptyset$) is defined to be $S = M \cup_{\partial} M$. For example in SAQ12 you saw that K^2, the Klein bottle, is the double of the Möbius band M^2:

$$K^2 = M^2 \cup_{\partial} M^2.$$

(a) Show by means of a sketch that

$$S^2 = D^2 \cup_{\partial} D^2$$

(a sphere is the double of a disc).

(b) What is the double of a cylinder?

(c) What is the double of a disc with three holes?

(d) What is the double of a torus with an open disc removed?

(e) Prove that the double of a non-orientable surface is non-orientable.

(f)* Show how to find a system of edge equations for the double of the surface represented by the edge equation A = 1.

[Hint: Express A as a sequence of blocks separating boundary edges.]

Section 4.1

Reduce each of the following edge equations to canonical form.

1. $abacbd = 1$
2. $aba^{-1}cb^{-1}c^{-1} = 1$
3. $ababcdc^{-1}d^{-1} = 1$
4. $abacbc = 1$
5. $abcbcda\ d^{-1} = 1$
6. $abbccddaef = 1$
7. $aba^{-1}b^{-1}cded^{-1}e^{-1}f = 1$
8. $aba^{-1}b^{-1}chc^{-1}ded^{-1}e^{-1}gkg^{-1} = 1$

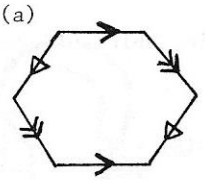
Reduce each of the following systems of edge equations to a single edge equation, and then find its canonical form.

9. $abc = 1$; $bfg = 1$, $gjh = 1$; $fed = 1$.
10. $abc^{-1}d^{-1} = 1$, $cde^{-1}f^{-1} = 1$; $efa^{-1}b^{-1} = 1$.

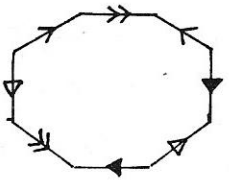
Section 4.2

1. In each of the following polygons with edge identifications, label vertices which are to be identified.

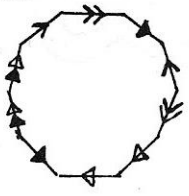
(a)



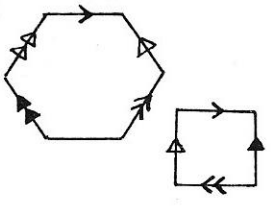
(b)



(c)



(d)



2. For each of the following systems of edge equations, determine the number of distinct vertices after identifications. Arrange any boundary edges into cyclic sets.
   (a) $abcd=1$; $bcf=1$; $dfgh=1$; $gkh=1$.
   (b) $abc=1$; $bgg=1$; $afk=1$; $fde=1$.
   (c) $abc=1$; $bfg=1$; $gjh=1$; $fed=1$.
   (d) $abc^{-1}d^{-1}=1$; $cde^{-1}f^{-1}=1$; $efa^{-1}b^{-1}=1$.

3. For each of the systems in Exercise 2, calculate χ and β and determine whether or not the surface in question is orientable. Write down the canonical form of each.

4. Compute χ and β for each of the surfaces represented below. State whether or not each is orientable (giving a brief reason).
   (a) $abacbded^{-1}e^{-1} = 1$
   (b) $aabcb^{-1}cdefe^{-1} = 1$
   (c) $abacbd = 1$, $defe^{-1}g = 1$
   (d) $abcb^{-1}a^{-1}de^{-1}efe^{-1} = 1$

5. Classify all the surfaces in Exercise 4.

6. Prove that the double of an orientable surface is orientable (cf. Exercise 1 on Section 3.3).

Section 4.3

1. SAQ12 of Unit 4 showed that one possible edge equation for $L \# M$ (where L, M are represented by $A = 1$, $B = 1$, respectively) is $AwBw^{-1} = 1$.

Show that $AwB^{-1}w^{-1} = 1$ is just as good. Deduce that in the special cases when M is a disc, projective plane or torus (and B is in canonical form) an edge equation for $L \# M$ is $AB^{-1} = 1$. Interpret this fact geometrically.

2. In SAQs 12–15 of Unit 4 we saw that if L is a surface with edge equation A = 1 and M is a disc, a projective plane, or a torus, with edge equation B = 1 in canonical form, then a possible edge equation for $L \# M$ is AB = 1. Deduce that this holds good if M is <u>any</u> surface with edge equation B = 1 in canonical form.

3.* Prove that, provided we keep clear of the boundaries of L and M, the definition of $L \# M$ is independent of which open discs are removed form L and M.

[Hint: Argue in stages as follows.

(i) Prove that removing a disc from the interior leaves a surface which can be represented by an edge equation

$$Aded^{-1} = 1.$$

(ii) Prove that if the disc meets an edge a of the polygon then we can "slide it off" provided that a is identified with some other edge.]

4. In SAQ16 of Unit 4 we showed that for surfaces L, M, N with edge equations A = 1, B = 1, AB = 1, respectively,

$$\chi(N) = \chi(L) + \chi(M) - 2.$$

Deduce that for any two surfaces L, M,

$$\chi(L \# M) = \chi(L) + \chi(M) - 2.$$

Hence compute:

(a) $\chi(pT^2 \# rD^2)$;

(b) $\chi(q\,\mathbb{R}P^2 \# rD^2)$.

5. Prove geometrically that

$$\chi(L \# M) = \chi(L) + \chi(M) - 2$$

by considering the definition of χ in the context of Frame 9.

6. Prove that $\beta(L \# M) = \beta(L) + \beta(M)$.

7. Use the classification theorem to decide which of the following pairs are homeomorphic, and briefly state your reasons.

(a) $D^2 \# \mathbb{R}P^2$, M^2.

(b) $T^2 \# M^2$, $K^2 \# M^2$.

(c) $M^2 \# M^2$, $M^2 \# D^2$.

(d) $K^2 \# D^2$, $K^2 \# M^2$.

(e) $2\,\mathbb{R}P^2 \# D^2$, $K^2 \# M^2$.

(f) $2\,\mathbb{R}P^2 \# D^2$, $\mathbb{R}P^2 \# D^2$.

(g) $\mathbb{R}P^2 \# D^2 \# M^2$, $2\,\mathbb{R}P^2 \# 2D^2$.

8. Are there any orientable surfaces S such that

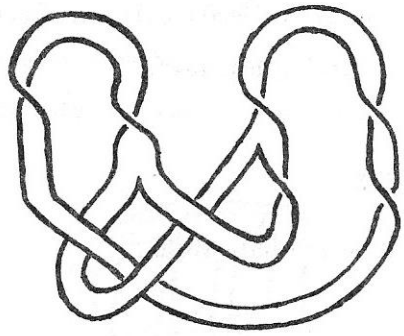
$$\chi(S) = -1,\ \beta(S) = 3?$$

What about $\chi(S) = -1$, $\beta(S) = 2$?

Miscellaneous Examples and Exercises on Units 1-4

Example

How would you analyse the following attractive surface?

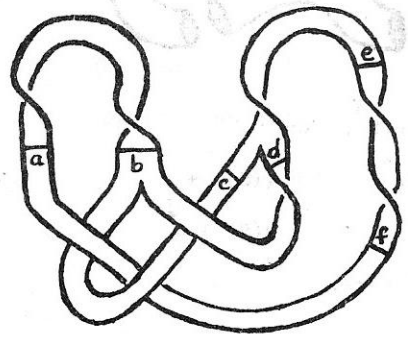


Solution

The simplest task is to find the number of boundary curves, for which tracing with coloured pens is both accurate and aesthetic. In this case the number is 2.

The next task is to compute its Euler characteristic, for which a subdivision into polygons is advised.

We have chosen the following one, which introduces the 6 edges a, b, c, d, e, f and consequently 12 vertices (not shown).

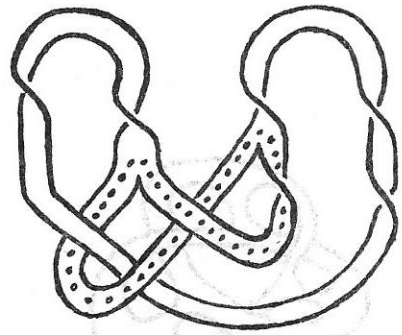


This divides one boundary curve into 5 pieces, the other into 7; so $E = 6 + 5 + 7 = 18$. The surface is divided into five faces. (Shade them as you count to be on the safe side.) So

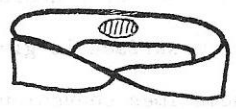
$\chi = F - E + V$

$= 5 - 18 + 12 = -1.$

The hardest task is to find whether or not it is orientable. The mechanical method is to cut it up into five polygons as above, but it is more fun to go Möbius band hunting. We have drawn one in by means of the dotted line.

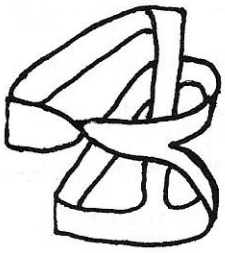


This surface is not orientable. Thus it is homeomorphic to a Möbius band with an open disc removed.

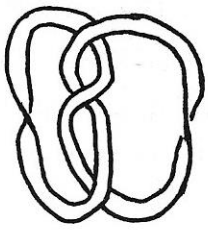


1. Use the classification theorem to describe each of the following in simple terms.

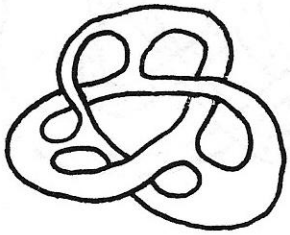
(a)



(b)



(c)



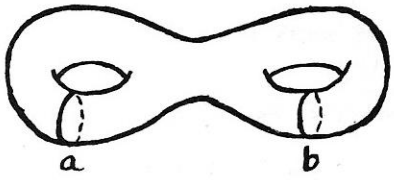
We have seen that adding a handle to a surface S (i.e. forming S # T^2) decreases the Euler characteristic by 2. When a surface is given pictorially it is sometimes convenient to analyse it by reversing this process. In other words, cut along a circle which goes round a handle, and attach discs to the boundary curves so formed. That is, replace

by

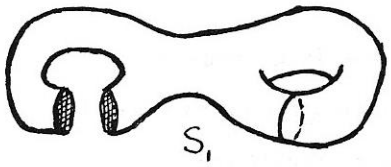
This raises the Euler characteristic by 2. We shall call this technique "cutting handles".

Example

Let S be a double torus.

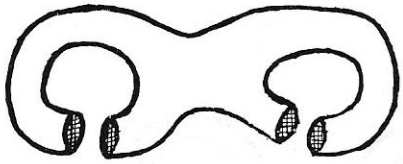


Cutting the handle at a gives



and $\chi(S_1) = \chi(S) + 2$.

Cutting the handle at b gives



a sphere S^2. Now

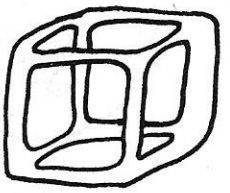
$$\chi(S^2) = 2 = \chi(S_1) + 2$$
$$= \chi(S) + 4.$$

So $\chi(S) = -2$.

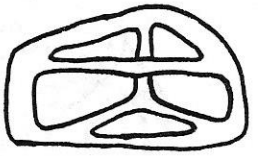
This confirms our previous knowledge, but it is a useful technique because it works even if the cutting process disconnects the surface altogether.

2. Find the Euler characteristics of each of the following surfaces, marking clearly each handle you cut. (All these are made of tubular material.)

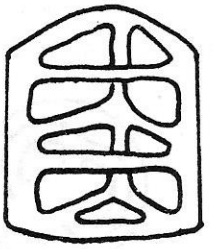
(a)



(b)

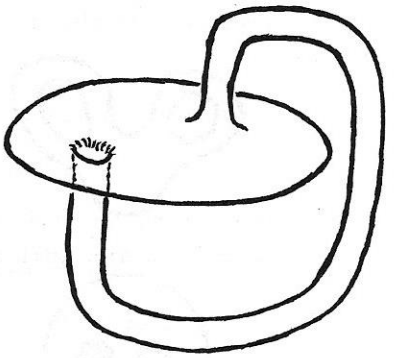


(c)



(d) The same kind of surface as (b) and (c), but with n layers.

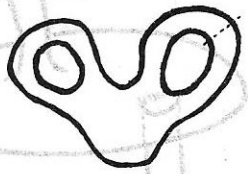
(e) Confirm, by cutting the handle, that this surface



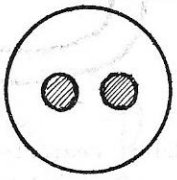
has Euler characteristic -1.

Section 1.1

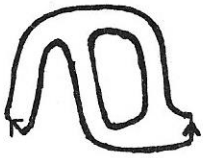
1. (a) Cut along the line, unlink the piece and re-glue to obtain



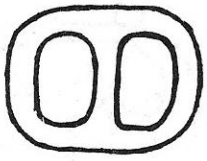
which is a disc with two holes.



(b) Cutting, unravelling

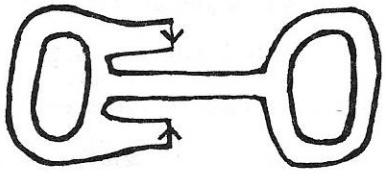


and re-glueing

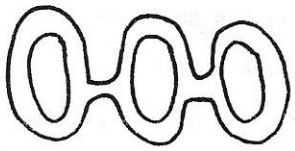


again yields a disc with two holes.

(c) Cutting and unpicking the free arms gives



Moving the free arms to the left, and re-glueing, we obtain



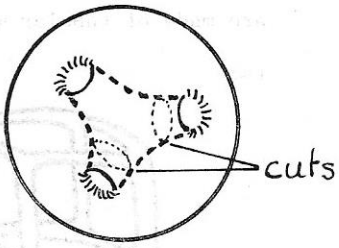
a disc with three holes.

(d) Cutting and unravelling gives

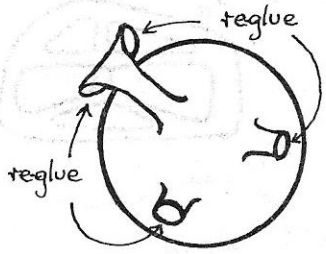
Re-glueing the free arm gives a disc with four holes.

2. (a) A torus. (Squash the picture vertically.)

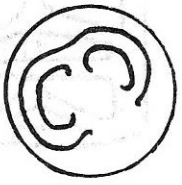
(b) A double torus. To see this, make two cuts as shown



and pull the "tubes" outside the sphere



Now re-glue the cuts to obtain

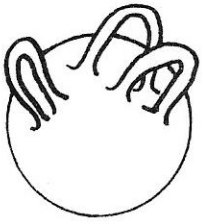


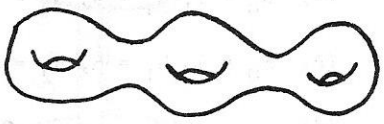
"Deflating" the sphere gives a double torus.

(c) A torus: cut, undo the knot, and re-glue.

(d) As above, a double torus.

(e) A triple torus. Slide the top handle down the others onto the sphere.





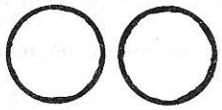
Now deflating the sphere yields a triple torus.

3. (a)

(three triangles)

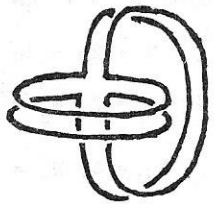
(b) None

(c)



(two circles)

(d)



(a circle)

4. (a) 3 (b) 0 (c) 2 (d) 1

Section 1.3

1.

	Path-connected	Closed	Bounded
(a)	✓	✓	X
(b)	✓	X	✓
(c)	✓	X	✓
(d)	✓	✓	✓
(e)	✓	✓	X

The important point here is that the annuli (b) and (c) are not closed. For example a point P(x, y) with $x^2 + y^2 = r$ does not lie in either, but any closed rectangle with P in its interior must meet both.

Section 1.4

1. (a) The vertex of the cone does not have a disc-like neighbourhood.

(b) P does not have a disc-like neighbourhood. Near P, the object looks like

This is the best "non-example" to remember!

(c) Again P does not have a disc-like neighbourhood. (The region near P looks very like the above picture.)

(d) Some points on the boundary of the cylinder have an interval-like neighbourhood (for example P below).

2. (a) This is a subdivision.

(b) Edges e and f do not meet in a vertex, so this is not a subdivision.

(c) This is not a subdivision, because:

(i) the edges do not form a path-connected network;

(ii) the face is not a topological disc.

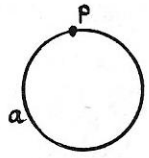
It becomes a subdivision when P and Q are joined by an edge.

(d) This is a subdivision.

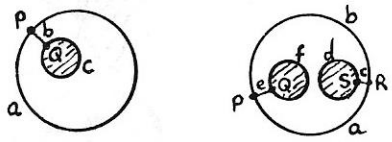
(e) This is a subdivision.

(f) This is not a subdivision because the edges do not meet in a vertex.

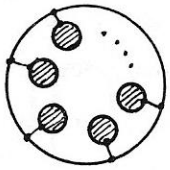
3. (a)



(b)

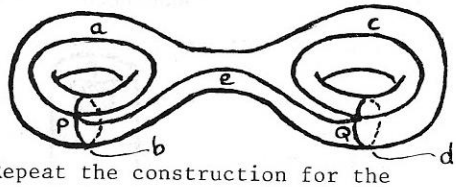


(c) Join the boundary of each hole to the boundary of the disc by a family of n non-intersecting edges.

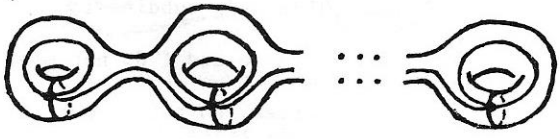


(d) See 2(d) above.

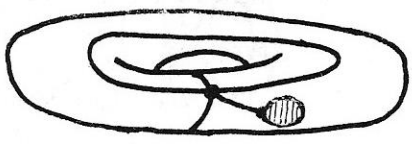
(e)



(f) Repeat the construction for the double torus.



(g)

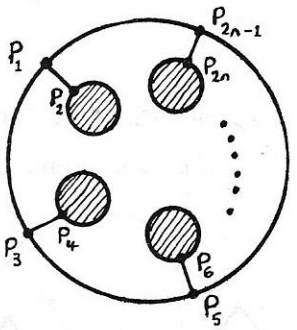


4. (a) $N_0 = 2$, $N_1 = 3$, $N_2 = 2$; $\chi = +1$.

(b) $N_0 = 2$, $N_1 = 3$, $N_2 = 1$; $\chi = 0$.

(c) $N_0 = 4$, $N_1 = 6$, $N_2 = 1$; $\chi = -1$.

(d)



It is clear that $N_0 = 2n$. There are n arcs on the outer boundary of the disc, n arcs bounding the holes, and n arcs connecting holes to outer boundary. Hence $N_1 = 3n$.

As before $N_2 = 1$, so

$$\chi = 1 + 2n - 3n = 1 - n.$$

(e) $N_0 = 1$, $N_1 = 2$, $N_2 = 1$; $\chi = 0$.

(f) $N_0 = 2$, $N_1 = 4$, $N_2 = 1$; $\chi = -1$.

(g) Using the subdivision of 3(f) above,

$N_0=N$, $N_1=3N-1$, $N_2=1$, $\chi=2-2N$.

5. Exercise 1 (Section 1.1)

(a) $\chi = -1$ (b) $\chi = -1$ (c) $\chi = -2$

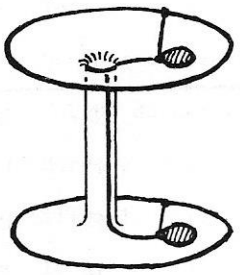
(d) $\chi = -3$

Exercise 2 (Section 1.1)

(a) $\chi = 0$ (b) $\chi = -2$ (c) $\chi = 0$

(d) $\chi = -2$ (e) $\chi = -4$

6.



$N_0 = 4$, $N_1 = 7$, $N_2 = 1$; $\chi = -2$.

7. Exercise 1 (Section 1.1)

(a) 3 (b) 3 (c) 4 (d) 5

Exercise 2 (Section 1.1)

$\beta = 0$ in all cases.

Exercise 4 (Section 1.4)

(a) 1 (b) 2 (c) 3 (d) $n + 1$

(e) 0 (f) 1 (g) 0

Section 2.1

1. (a) Let the number of faces, edges, vertices be F, E, V, respectively.

Then $F - E + V = -6$. (*)

In the usual way we have

$$kF = 2E, \quad jV = 2E$$

and so

$$\left(\frac{2}{k} - 1 + \frac{2}{j}\right) E = -6.$$

Since $E > 0$ we have

$$\frac{2}{k} + \frac{2}{j} - 1 < 0, \text{ or } \frac{1}{k} + \frac{1}{j} < \frac{1}{2}.$$

(b) For a regular triangulation, $k = 3$.

Thus $3F = 2E$,

From (*) we obtain

$$V - \frac{F}{2} = -6,$$

or $F = 2V + 12$.

Thus $V = \frac{2}{j} E$

$$= \frac{3F}{j}$$

$$= \frac{6V + 36}{j}.$$

Hence $j = 6 + \frac{36}{V}$, and since j is an integer, V must be a factor of 36.

$V = 1, 2, 3, 4, 6, 9, 12, 18$ or 36.

This means that $7 \leq j \leq 42$.

(c) If $j = 8$ then the above argument shows that $V = 18$.

Hence $E = \frac{jV}{2} = 72$, and $F = \frac{2E}{3} = 48$.

2. As in Exercise 1 we deduce that

$3F = 2E$, $F - E + V = -2$.

Then $V - \frac{F}{2} = -2$, or $F = 2V + 4$.

Rearranging gives $j = 6 + \frac{12}{V}$. Since j is an integer, V is a factor of 12 and the possibilities are:

V	F	E	j
1	6	9	18
2	8	12	12
3	10	15	10
4	12	18	9
6	16	24	8
12	28	42	7

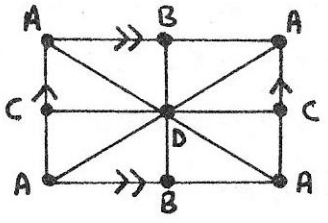
(Notice that most of these cannot in fact lead to genuine triangulation. For since we require that at most on edge joins any two given vertices, w must have

$$j \leq V - 1.$$

This rules out all except the last possibility.)

3. (a) This is a triangulation.

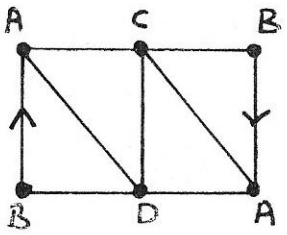
(b)



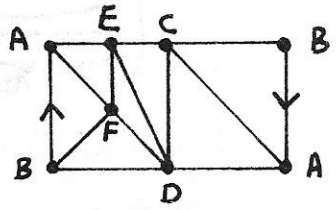
This is not a triangulation because the vertices {A, D} span more than one edge. So do {A, C}, {D, B}, {C, D}. It becomes a triangulation after one barycentric subdivision.

(c) This fails for exactly the same reasons as (b) above. The remedy is also the same.

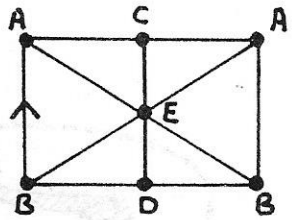
(c



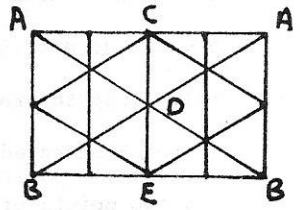
This fails to be a triangulation because the vertices {A, C}, {A, D} do not span a unique edge. Moreover there are two distinct triangles with vertices {A, D, C}. It can be made into a triangulation in several ways, for example:



(e)



This is not a triangulation because the vertices {A, C} do not span a unique edge. Nor do the pairs {A, D}, {A, E}, {B, D}, {B, E}. Also, there are two faces with vertices {A, C, E} and two with vertices {B, D, E}. There are various remedies, for example:



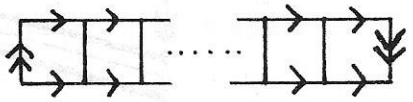
Section 2.3

1. (a) If j = k = 4 we may have a subdivision into N squares, with 2N edges and N vertices [4F=2E=4V]. If j = 6 and k = 3 we may have a subdivision with 2N triangles, with 3N edges and N vertices [3F = 2E = 6V].

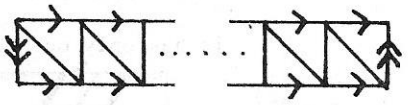
(b) All three possibilities are easily achieved.

j = 3, k = 6 refer to the first part of the solution to SAQ14 of Unit 2. (Instead of four hexagons, draw N.)

j = k = 4

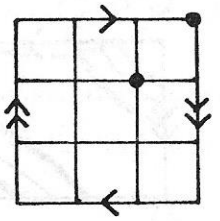


j = 6, k = 3



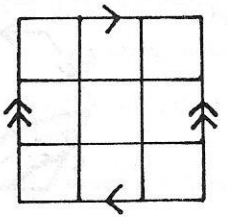
(Just add N edges to the above picture for squares.)

2. (a)



A corner vertex lies on only 2 edges, whilst an "interior" vertex lies on 4. Hence the subdivision cannot be regular.

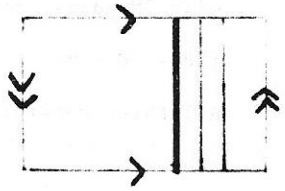
(b) Yes.



The above defect is now remedied, since all the corners are identified to a single vertex. All the conditions for a regular subdivision are now met.

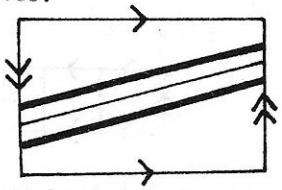
Section 2.4

1. (a)



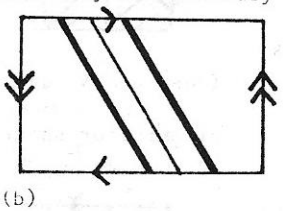
Neighbourhood is a cylinder because it has two boundary curves.

(b)



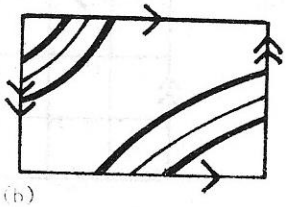
Neighbourhood is a Möbius band - it has only one boundary curve.

(c)



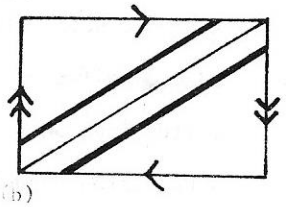
As (b)

(d)



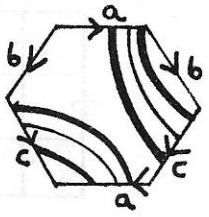
As (b)

(e)



As (b)

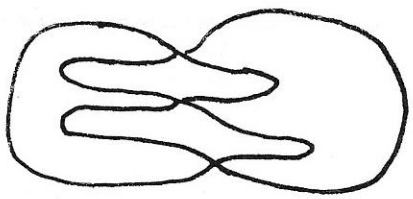
(f)



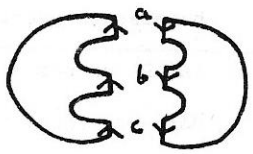
As (b)

2. (a) This is a Möbius band, so not orientable.

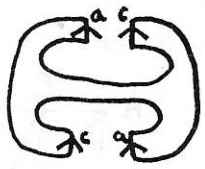
(b)



Cut along a, b, c to get

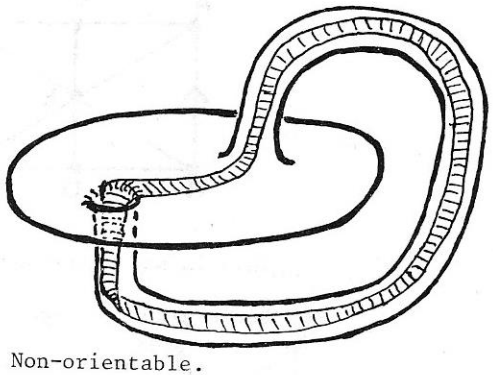


This can be redrawn as



which is two-sided and hence orientable.

(c)



Non-orientable.

(d) This is a surface without boundary embedded in $\mathbb{R}^3$, and is thus orientable.

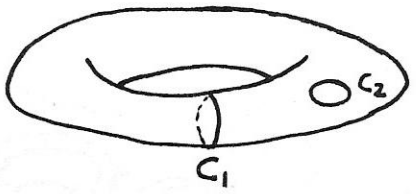
Solutions to the Miscellaneous Questions

1. (a) By $S \setminus C$ we mean those points of S which do not lie on C. It is sufficient to prove that $S \setminus C$ is path-connected. Let P, Q be any two points of $S \setminus C$. Draw a neighbourhood A of C which is small enough not to include P or Q.

Let σ be any path in S from P to Q, i.e. $\sigma:[0, 1] \to S$, $\sigma(0) = P$, $\sigma(1) = Q$.

If σ does not cross C we have finished. If it does, it must also cross C', the boundary curve of A. (There is only one boundary curve since C is orientation-reversing.) Now let $t_0 \in [0, 1]$ be the smallest number in [0, 1] such that $\sigma(t_0) \in C'$. Similarly, let $t_1 \in [0, 1]$ be the largest number in [0, 1] with $\sigma(t_1) \in C'$. Replace σ by the path σ' which agrees with σ on $[0, t_0] \cup [t_1, 1]$ and which runs around C' from $\sigma(t_0)$ to $\sigma(t_1)$. This is obviously continuous and is a path joining P to Q. It does not cross C.

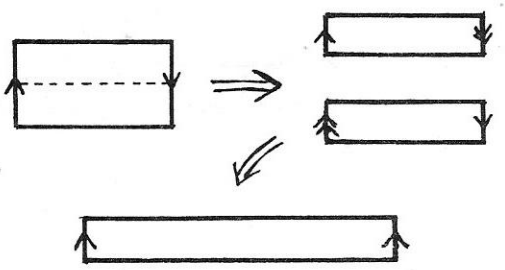
(b)



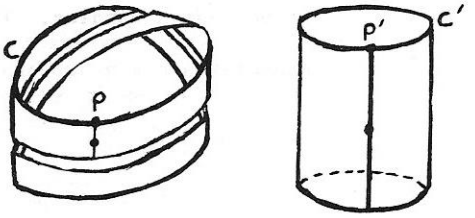
Cutting a torus along the orientation-preserving path C_1 leaves a path-connected cylinder. Cutting along C_2 gives two pieces.

2. (a) A cylinder (embedded in $\mathbb{R}^3$ with two half-twists).

(b)

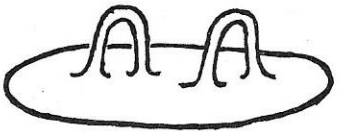


[The surface has two boundary curves, and its Euler characteristic has not changed, so χ (new surface) = 0. Going around it (on your model) does not affect orientation and so it is orientable. This suggests that the surface is a cylinder. In fact, once we have the cla i ation theorem in Unit 4, we can prove that it is. But in this case we can exhibit directly an obvious homeomorphism between the model and a cylinder.

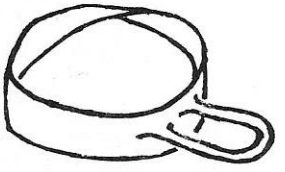


First map the boundary circle C of the Möbius band homeomorphically onto the top circle C' of the cylinder. Now observe that every point on the cut band is on a unique chord from some point on C which is "perpendicular" to C. Now extend the homeomorphism of C to C' to a homeomorphism of the cut band with the cylinder.]

3. A disc with any number of handles



or a Möbius band with any number of handles



all satisfy the requirements.

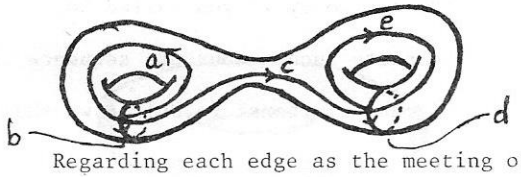
4. A cylinder with n handles has two boundary curves. So does the "sum" of two Möbius bands (as drawn in Frame 20 of Section 2.4 of Unit 2), and it does if you add n handles. These latter examples are non-orientable.

SOLUTIONS FOR UNIT 3

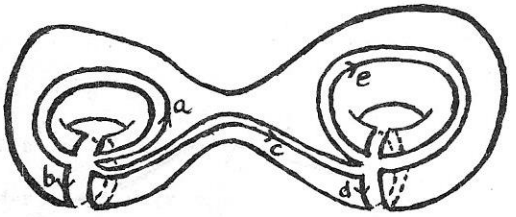
Section 3.1

1. In each case the figure is a singular topological polygon, but fails the surface test. For in each case there is a point which does not have a disc neighbourhood. At this point, two vertices of a polygon (or two different polygons) coincide, although this coincidence is not forced by the identification of edges. Condition (2) of the definition on Blackett page 58 is violated.

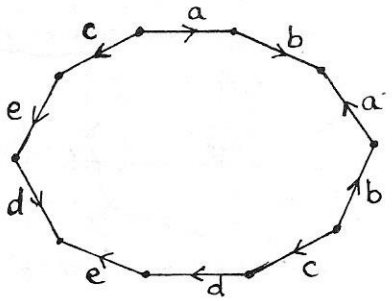
2. In fact, we can use the "standard" subdivision of the double torus.



Regarding each edge as the meeting of two sides of the decagon,



we can now label the decagon to ensure the correct identifications.



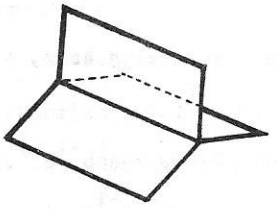
3. (a) T is not a vertex of the first rectangle nor is R a vertex of the second.

(b) The edges of the two rectangles do not meet in vertices.

(c) The two "rectangular" faces meet in an edge and in an isolated vertex.

(d) This is not a topological polygon because it is the image of a triangle all three of whose edges coincide, whereas edges occur at most twice in a topological polygon.

As a matter of fact, it cannot be a surface either, since a neighbourhood of any interior point of an edge of the triangle would look like



4. (a) This is a valid edge equation.

(b) This is not an equation.

(c) This is not valid - the symbol a occurs more than twice.

(d) Again this is not valid since the symbol a occurs more than twice.

(e) This is a valid edge equation.

(f) This is a valid edge equation.

5. (a) $abca^{-1}b^{-1}c^{-1} = 1$

(b) $aba^{-1}cb^{-1}c^{-1} = 1$

(c) $aa^{-1}bb^{-1}cc^{-1} = 1$

(d) $aba^{-1}db^{-1}c = 1$

6. (a) $xab^{-1}ca = 1$; $x^{-1}cbdd = 1$.

Writing the second as $x = cbdd$, and substituting for x in the first gives

$$cbddab^{-1}ca = 1.$$

(b) $xbab = 1$; $x^{-1}cc^{-1}d^{-1} = 1$.

The same procedure yields

$$cc^{-1}d^{-1}bab = 1.$$

(c) $xb^{-1}af^{-1}d^{-1}a^{-1} = 1$; $x^{-1}g^{-1}gf^{-1} = 1$.

As above, $g^{-1}gf^{-1}b^{-1}af^{-1}d^{-1}a^{-1} = 1$.

(d) $xabcde = 1$; $x^{-1}ab^{-1}c^{-1}d^{-1}e^{-1} = 1$.

Again, $ab^{-1}c^{-1}d^{-1}e^{-1}abcde = 1$.

7. (a) Rewrite $abacb^{-1}c^{-1} = 1$, starting at the second a.

(b) $abacb^{-1}c^{-1} = 1 \longrightarrow abacb^{-1} = c$

$\longrightarrow abac = cb$

$\longrightarrow aba = cbc^{-1}$

(c) Relabel a as x, b as y and c as z^{-1}.

(d) Put $y = acb^{-1}c^{-1}$. Then

$abacb^{-1}c^{-1} = 1 \longrightarrow aby=1$; $y=acb^{-1}c^{-1}$.

Section 3.2

1. $abcba=1 \longrightarrow aabcb=1$ (cycling)

$\longrightarrow aayb=1$; $y=bc$

$\longrightarrow aayyc^{-1}=1$

$\longrightarrow aabbc^{-1}=1$

2. $Kabcbah=1 \longrightarrow Kdcbah=1$; $ab=d$

$\longrightarrow Kdca^{-1}dah=1$

$\longrightarrow Kedah=1$; $e=dca^{-1}$

$\longrightarrow Keeac^{-1}ah=1$

$\longrightarrow Keefah=1$; $f=ac^{-1}$

$\longrightarrow Keeffch=1$

3. $abcb=1 \longrightarrow adb=1$; $d=bc$

$\longrightarrow addc^{-1}=1$

$\longrightarrow ddc^{-1}a=1$

$\longrightarrow bba^{-1}c=1$

4. $abcacb=1 \longrightarrow azb=1$; $z=bcac$

$\longrightarrow azb=1$; $b=zc^{-1}a^{-1}c^{-1}$

$\longrightarrow azzc^{-1}a^{-1}c^{-1}=1$

$\longrightarrow zzc^{-1}a^{-1}c^{-1}a=1$

$\longrightarrow zzc^{-1}wa=1$; $w=a^{-1}c^{-1}$

$\longrightarrow zzc^{-1}wa=1$; $c^{-1}=aw$

$\longrightarrow zzawwa=1$

$\longrightarrow zzay=1$; $y=wwa$

$\longrightarrow zzay=1$; $a=w^{-1}w^{-1}y$

$\longrightarrow zzw^{-1}w^{-1}yy=1$

$\longrightarrow aabbcc=1$

(Do not worry if you failed to complete such an obscure sequence of manipulations: in Unit 4 we shall develope a foolproof and systematic procedure.)

5. $acbca^{-1}b^{-1}=1 \longrightarrow ayca^{-1}b^{-1}=1$; $y=cb$

$\longrightarrow ayyb^{-1}a^{-1}b^{-1}=1$

$\longrightarrow yyb^{-1}a^{-1}b^{-1}a=1$

$\longrightarrow yyzb^{-1}a=1$; $z=b^{-1}a^{-1}$

$\longrightarrow yyzzaa=1$

$\longrightarrow aabbcc=1$

(The same comment applies as in the solution above.)

6. $axcyx^{-1}y^{-1}F=1 \longrightarrow axdx^{-1}y^{-1}F=1$; $d=cy$

$\longrightarrow axdx^{-1}d^{-1}cF=1$

$\longrightarrow axyx^{-1}y^{-1}cF=1$

7. $axcydx^{-1}y^{-1}F=1 \longrightarrow axedx^{-1}y^{-1}F=1;\ e=cy$

$\longrightarrow axedx^{-1}e^{-1}cF=1$

$\longrightarrow axewe^{-1}cF=1;\ w=dx^{-1}$

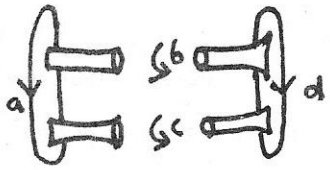
$\longrightarrow aw^{-1}dewe^{-1}cF=1$

$\longrightarrow aw^{-1}ewe^{-1}dcF=1$

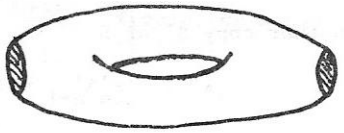
(using Exercise 6)

$\longrightarrow axyx^{-1}y^{-1}dcF=1$

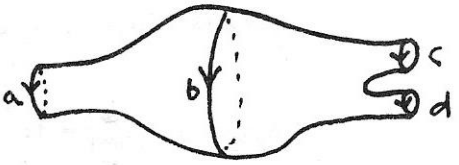
8. Pushing the b's and c's towards one another, we obtain



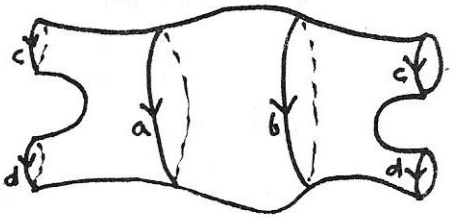
which is clearly a torus with two open discs removed.



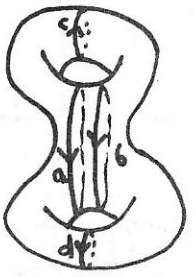
9. There are many ways to start. We first glue the b edges to get



Now glue the a's together.



Glueing the c's and d's now gives a double torus.



10. (a) is not orientable since a and b both occur twice in the same sense. c and d are boundary edges.

(b) is not orientable for the same reason. There are no boundary edges.

(c) is not orientable because a occurs twice in the same sense. c^{-1} and d are boundary edges.

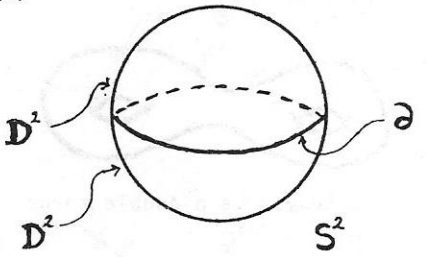
(d) is not orientable because b and c occur twice in the same sense. There are no boundary edges.

(e) is orientable, and c, f are boundary edges.

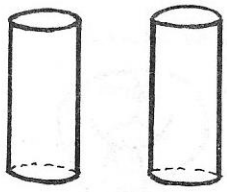
(f) is also orientable, and d, h are boundary edges.

Section 3.3

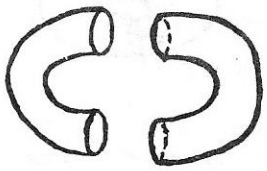
1. (a)



(b) A torus.

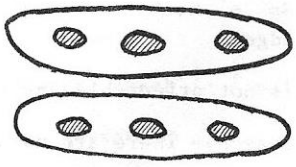


Bend the cylinders round,



and join up the boundaries!

(c)

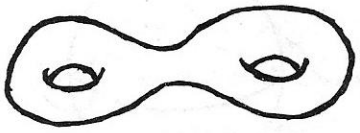


The easiest way to see this is to deform the edges into "tubes" and then join corresponding tubes.

The result is a triple torus.

(d)

Again, deform the edges into tubes and join up.



The result is a double torus.

(e) If S is non-orientable, then S contains a Möbius band. Since forming the double of S merely adds in more points, the double of S still contains the original Möbius band. Hence it is non-orientable.

(f) Following the hint, write A in the form

$$A_1 x_1 A_2 x_2 \ldots A_n x_n A_{n+1}$$

where $x_1, x_2, \ldots, x_n$ are all the boundary edges. (Thus any symbol occurring in an A_i occurs again in some A_j.)

Then an edge equation for S is

$$A_1 x_1 A_2 x_2 \ldots a_n x_n A_{n+1} = 1$$

and by adding dashes we obtain another copy S' of S

$$A'_1 x'_1 A'_2 x'_2 \ldots A'_n x'_n A'_{n+1} = 1.$$

Dropping the dashes from the x'_i's is equivalent to glueing S to S' along their boundaries. But this gives the double of S. Hence a suitable system of edge equations is

$$A_1 x_1 A_2 x_2 \ldots A_n x_n A_{n+1} = 1;$$

$$A'_1 x_1 A'_2 x_2 \ldots A'_n x_n A'_{n+1} = 1.$$

SOLUTIONS FOR UNIT 4

Section 4.1

1. $abacbd=1 \xrightarrow{\text{Lemma 1}} aab^{-1}cbd=1$

$\longrightarrow aab^{-1}cbdee^{-1}=1$

$\longrightarrow e^{-1}aab^{-1}cbde=1$

$\xrightarrow{\text{Lemma 1}} aeab^{-1}cbde=1$

$\xrightarrow{\text{Lemma 1}} aae^{-1}b^{-1}cbde=1$

$\xrightarrow{\text{Lemma 2}} aab^{-1}cbe^{-1}de=1$

(This is a Möbius band with a hole in it.)

2. $aba^{-1}cb^{-1}c^{-1}=1 \longrightarrow c^{-1}aba^{-1}cb^{-1}=1$

$\longrightarrow xbx^{-1}b^{-1}=1$

(re-labelling $c^{-1}a$ as x)

3. $ababcdc^{-1}d^{-1}=1 \longrightarrow xxcdc^{-1}d^{-1}=1$

(re-labelling ab as x)

$\longrightarrow xxyyzz=1$

(replacing a handle by two crosscaps)

4. $abacbc=1 \xrightarrow{\text{Lemma 1}} aab^{-1}cbc=1$

$\xrightarrow{\text{Lemma 1}} aab^{-1}b^{-1}cc=1$

$\longrightarrow aabbcc=1$

(re-labelling)

5. $abcbcda\ d^{-1}=1 \xrightarrow{\text{Lemma 1}} abbc^{-1}cda\ d^{-1}=1$

$\longrightarrow abbdad^{-1}=1$

$\longrightarrow bbdad^{-1}a=1$

$\xrightarrow{\text{Lemma 1}} bbddaa=1$

6. $abbccddaef=1 \longrightarrow abbccddag=1$

(re-labelling g for ef)

$\xrightarrow{\text{Lemma 1}} aad^{-1}d^{-1}c^{-1}c^{-1}b^{-1}b^{-1}g=1$

$\longrightarrow aaxxyyzzege^{-1}=1$

(re-labelling, and writing the boundary curve g as a cuff ege^{-1})

7. $aba^{-1}b^{-1}cded^{-1}e^{-1}f=1 \longrightarrow aba^{-1}b^{-1}ded^{-1}ce^{-1}f=1$

$\longrightarrow aba^{-1}b^{-1}decd^{-1}e^{-1}f=1$

$\longrightarrow aba^{-1}b^{-1}dced^{-1}e^{-1}f=1$

$\longrightarrow aba^{-1}b^{-1}ded^{-1}e^{-1}cf=1$

(all the above moves by Lemma 2)

$\longrightarrow aba^{-1}b^{-1}ded^{-1}e^{-1}g=1$

$\longrightarrow aba^{-1}b^{-1}ded^{-1}e^{-1}hgh^{-1}=1$

(writing the boundary curve g as a cuff)

8. $aba^{-1}b^{-1}chc^{-1}ded^{-1}e^{-1}gkg^{-1}=1$

$\longrightarrow aba^{-1}b^{-1}ded^{-1}e^{-1}chc^{-1}gkg^{-1}=1$

(by Step 2)

9. $abc=1;\ bfg=1;\ gjh=1;\ fed=1$

$\longrightarrow a^{-1}c^{-1}fg=1;\ gjh=1;\ fed=1$

$\longrightarrow a^{-1}c^{-1}fh^{-1}j^{-1}=1;\ fed=1$

$\longrightarrow a^{-1}c^{-1}d^{-1}e^{-1}h^{-1}j^{-1}=1$

$\longrightarrow axa^{-1}=1$

10. $abc^{-1}d^{-1}=1;\ cde^{-1}f^{-1}=1;\ efa^{-1}b^{-1}=1$

$\longrightarrow abde^{-1}f^{-1}d^{-1}=1;\ efa^{-1}b^{-1}=1$

$\longrightarrow abdfa^{-1}b^{-1}f^{-1}d^{-1}=1$

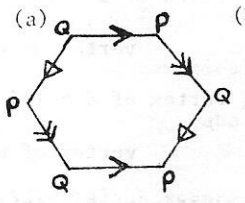
$\longrightarrow adfba^{-1}b^{-1}f^{-1}d^{-1}=1$

$\longrightarrow aba^{-1}b^{-1}dff^{-1}d^{-1}=1$

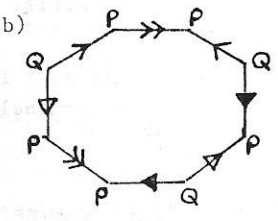
$\longrightarrow aba^{-1}b^{-1}=1$

Section 4.2

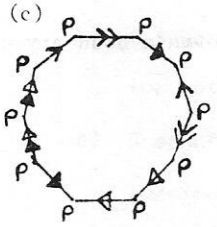
1. (a)



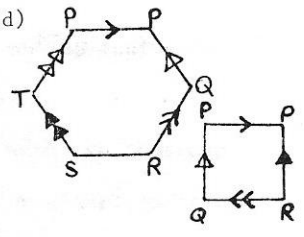
(b)



(c)



(d)



2. (a) We start with (ab) and identify as many vertices as possible:

$(ab)=(fb)=(fg)=(hg)=(hd)=(cd)$
$=(cf)=(df)=(da)$.

Of the five remaining vertices, (bc) clearly forms a sequence on its own (and occurs twice); $(gh)=(kh)=(gk)$.

Clearly, the boundary edges are a, k. Moreover, the initial vertex of a is also the terminal vertex of a, and similarly for k. Thus a and k are both boundary curves.

(b) $(ab)=(af)=(ef)$

and continuing the other way from (ab) gives

$(ab)=(gb)=(gg)=(bg)=(bc)$.

An unused vertex is (ca):

$(ca)=(ka)$ and the sequence terminates. Another is (fk):

$(fk)=(fd)$ and again the sequence stops. The last remaining vertex is (de). The boundary edges are c, k, d, e.

Now terminal vertex of c = terminal vertex of k,
initial vertex of k = initial vertex of d,
terminal vertex of d = initial vertex of e,
terminal vertex of e = initial vertex of c.

Thus $ck^{-1}de$ forms a boundary curve.

(c) $(ab)=(gb)=(gj)$
$(bc)=(bf)=(df)$
(ca)
$(fg)=(hg)=(fe)$
(ed)
(jh)

Boundary edges are a, c, d, e, h, j.

Terminal vertex of a = initial vertex of j
Terminal vertex of j = initial vertex of h
Terminal vertex of h = initial vertex of e
Terminal vertex of e = initial vertex of d
Terminal vertex of d = initial vertex of c
Terminal vertex of c = initial vertex of a

Thus ajhedc forms a boundary curve.

(d) $(ab)=(fa^{-1})=(e^{-1}f^{-1})=(b^{-1}e)$
$(bc^{-1})=(cd)=(d^{-1}a)=(a^{-1}b^{-1})$
$(c^{-1}d^{-1})=(de^{-1})=(ef)=(f^{-1}c)$

There are no boundary edges.

3. (a) $N_0=3$, $N_1=8$, $N_2=4$.

$\chi=-1$, and we have seen that $\beta=2$.

Examining the first three equations, $abcd=1$, $bcf=1$, $dfgh=1$, we see that for orientability one of the first two must be inverted. Similarly one of the second two must be (to get rid of f, f). But so must one of the outer two (to eliminate d, d). This is impossible, so the surface is non-orientable.

Its canonical form is

$$aabcb^{-1}ded^{-1}=1.$$

(b) $N_0=4$, $N_1=8$, $N_2=4$.

$\chi=0$, $\beta=1$.

Because of the second equation the surface is non-orientable.

Its canonical form is

$$aabcb^{-1}=1.$$

(c) $N_0=6$, $N_1=9$, $N_2=4$.

$\chi=1$, $\beta=1$.

Inverting the second equation satisfies the criterion for orientability. The canonical form is

$$aba^{-1}=1.$$

(d) $N_0=3$, $N_1=6$, $N_2=3$.

$\chi=0$, $\beta=0$.

The system satisfies the orientability condition as it stands. The canonical form is

$$aba^{-1}b^{-1}=1.$$

4. (a) $(ab)=(cb)=(ac)$

$(ba)=(bd)=(d^{-1}e^{-1})=(ed^{-1})$

$=(de)=(e^{-1}a)$

$N_0=2$, $N_1=5$, $N_2=1$.

$\chi=-2$.

c is the unique boundary curve, so $\beta=1$.

The surface is non-orientable due to the occurrence of a, a. Its canonical form is

$$aabbccded^{-1}=1.$$

(b) $(aa)=(ab)=(b^{-1}c)=(e^{-1}a)=(de)$

$(bc)=(cb^{-1})=(cd)$

$(ef)=(fe^{-1})$

$N_0=3$, $N_1=6$, $N_2=1$.

$\chi=-2$.

d and f are boundary edges, and each is a boundary curve, so $\beta = 2$.

The surface is non-orientable, due to the occurrence of a, a. Hence its canonical form is

$$aabbcdc^{-1}efe^{-1}=1.$$

(c) $(ab)=(ac)=(cb)$

$(ba)=(bd)=(gd)=(da)=(de)=(e^{-1}g)$

$(ef)=(fe^{-1})$

$N_0=3$, $N_1=7$, $N_2=2$.

$\chi=-2$.

c, f, g are boundary edges, and each is a boundary curve. Hence $\beta=3$.

The surface is non-orientable because of a, a, so the canonical form is

$$aabcb^{-1}ded^{-1}fgf^{-1}=1.$$

(d) $(ab)=(b^{-1}a^{-1})$

$(a^{-1}d)=(e^{-1}a)=(c^{-1}e)=(bc)=(cb^{-1})=(dc^{-1})$

$(ef)=(fe^{-1})$

$N_0=3$, $N_1=6$, $N_2=1$.

$\chi=-2$.

d and f are boundary edges and in fact boundary curves, so $\beta = 2$.

The surface is orientable, so its canonical form is

$$aba^{-1}b^{-1}cdc^{-1}efe^{-1}=1.$$

5. (a) A sphere with three crosscaps and a hole.

(b) A sphere with two crosscaps and two holes.

(c) A sphere with one crosscap (projective plane) and three holes.

(d) A torus with two holes.

6. Recall that if a surface S has an edge equation A=1, then we can "factorize" A into the form

$$A_1x_1A_2x_2\ldots A_nx_nA_{n+1}=1$$

where the x_i are all the boundary edges. Thus any symbol occurring in an A_i occurs also in some A_j. Now if S is orientable, then each such second occurrence is of opposite sign. Thus in the system of equations which represents the double of S

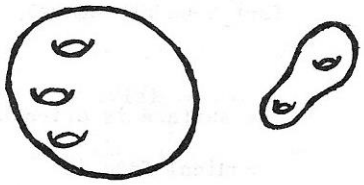
$$A_1x_1A_2x_2\ldots A_nx_nA_{n+1}=1;$$

$$A'_1x_1A'_2x_2\ldots A'_nx_nA'_{n+1}=1$$

we may invert the second equation and then every symbol in the resulting system occurs twice with opposite sign. Thus the double of S is orientable if S is.

Section 4.3

1. Let L, M be represented by A=1, B=1, respectively.



Remove an open disc from each. This can be done by drawing an edge from the last vertex of A and the first vertex of B to obtain edge equations

$$Abcb^{-1}=1$$

$$ded^{-1}B=1.$$

Now glue the cylinder $xe^{-1}x^{-1}c=1$ to the boundaries of the holes. The resulting surface is L#M and is represented by the system $Abcb^{-1}=1$; $xe^{-1}x^{-1}c=1$; $ded^{-1}B=1$. Eliminating c between the first and second gives

$$b^{-1}Abxex^{-1}=1;\ ded^{-1}B=1.$$

Eliminating e leaves the single equation $b^{-1}Abxd^{-1}B^{-1}dx^{-1}=1$.

By Lemma 2, move x around $d^{-1}B^{-1}d$ to cancel x^{-1}:

$$b^{-1}Abd^{-1}B^{-1}d=1.$$

Now move d around $b^{-1}Ab$ to cancel d^{-1}:

$$b^{-1}AbB^{-1}=1,$$

or (cycling)

$$AbB^{-1}b^{-1}=1, \text{ as required.}$$

The remaining deduction is immediate from SAQs 13-15.

Geometrically, all that has happened is that instead of joining L and M "from the outside", reversing the orientation of one of the boundaries glues M "inside" L (or vice versa).

This does not change the homeomorphism type of L#M.

2. If M is represented by B=1 in canonical form, then

$$B=\begin{cases} a_1a_1\ldots a_qa_qd_1e_1d_1^{-1}\ldots d_re_rd_r^{-1} \\ a_1b_1a_1^{-1}b_1^{-1}\ldots a_pb_pa_p^{-1}b_p^{-1}d_1e_1d_1^{-1}\ldots d_re_rd_r \end{cases}$$

accordingly as M is non-orientable or orientable.

Now an edge equation for L#M is $AwBw^{-1}=1$ which appears as

$$Awa_1a_1\ldots a_qa_qd_1e_1d_1^{-1}\ldots d_re_rd_r^{-1}w^{-1}=1$$

when M is non-orientable (and a corresponding form when M is orientable).

First move w past each crosscap, then past each cuff until it cancels w^{-1}.

The resulting equation is

$$AB=1.$$

(An analogous argument works in the orientable case.)

Notice that in general it is NOT true that if L, M are represented by edge equations A=1, B=1, respectively, then L#M is represented by AB=1.

For example if L and M are the Möbius bands

$$A=aae=1$$

$$B=bbf=1$$

then L#M must have two boundary curves. But

$$AB=aaebbf=1$$

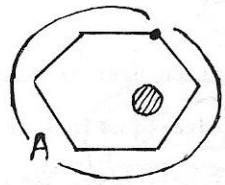
$$\longrightarrow aabbef=1$$

$$\longrightarrow aabbg=1$$

has only one!

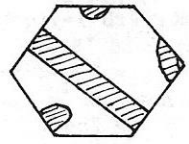
3. We confine our attention to showing that the definition of L#M is independent of which disc is removed from L (since the same argument will work for M).

Suppose then that a representation of L as a polygon with edge equation A=1 has been agreed.



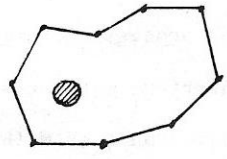
If we remove a disc from the interior of the polygon, then it is certainly possible to describe the resulting surface in the form $Aed^{-1}=1$.

The argument leading to an edge equation for L#M now proceeds unhindered. So suppose that the disc we remove crosses some edges of the polygon.



All we do now is to cut the polygon up in such a way that all different regions of the disc are separated, and re-glue so that the disc is now in the interior of the new polygon.

The only problem about this is that if the disc meets a boundary (i.e. non-identified) edge e then we are unable to proceed. Having arranged for the disc to be in the interior we can now shrink it until it is small.



Now adjust its position until the original cut and glue operations can all be undone without meeting the disc. We are now back to the first case.

4. The deduction part is easy. Express L, M as edge equations in canonical form, say A=1, B=1. Then L#M is represented by AB=1 and we can use the information given.

(a) $\chi(pT^2\#rD^2)$

$= \chi(pT^2)+\chi(rD^2)-2$

$=[\chi(T^2)+\chi(p-1)T^2)-2]$

$+[\chi(D^2)+\chi((r-1)D^2)-2]-2$

$= -2(p-1)+r-2(r-1)-2$

$= 2-2p+2-r-2$

$= 2-2p-r$

(b) $\chi(q\mathbb{R}P^2\#rD^2)=\chi(q\mathbb{R}P^2)+\chi(rD^2)-2$

$= 2-q+2-r-2$

$= 2-q-r$

5. We may suppose that L and M are triangulated in such a way that L#M is formed by removing a triangular face from each, and glueing a triangular-section cylinder between them. Adding the cylinder adds three faces and three edges and so contributes nothing to the Euler characteristic. However, the original removal of two faces reduces χ by 2.

6. The process of forming L#M cannot interfere with existing boundary curves on L or M (because the construction specifically excludes the boundaries).

Since no new boundary curves are created either, we have

$$\beta(L\#M)=\beta(L)+\beta(M).$$

7.

	Surface	χ	β	orientable	
(i)	$D^2\#\mathbb{R}P^2$	0	1	x	
	M^2	0	1	x	homeomorphic
(ii)	$T^2\#M^2$	-2	1	x	
	$K^2\#M^2$	-2	1	x	homeomorphic
(iii)	$M^2\#M^2$	-2			
	$M^2\#D^2$	-1			not homeomorphic
(iv)	$K^2\#D^2$	-1			
	$K^2\#M^2$	-2			not homeomorphic
(v)	$2\mathbb{R}P^2\#D^2$	-1			
	$K^2\#M^2$	-2			not homeomorphic
(vi)	$2\mathbb{R}P^2\#D^2$	-1			
	$\mathbb{R}P^2\#D^2$	0			not homeomorphic
(vii)	$\mathbb{R}P^2\#D^2\#M^2$	-2	2	x	
	$2\mathbb{R}P^2\#2D^2$	-2	2	x	homeomorphic

8. An orientable surface S without boundary has Euler characteristic 2-2g, say. So $S\#rD^2$ has Euler characteristic

$$2-2g-r,$$

and $\beta(S\#rD^2)=r$.

Solving for g, r we have

$$2-2g-r=-1$$

$$r=3$$

or $2-2g=2$

i.e. g=0, r=3.

a sphere with three holes. However for

$$2-2g-r=-1$$

$$r=2$$

we obtain

$2-2g=1$

which is not solvable since g must be an integer. There is, however, a non-orientable surface which fits this description. The corresponding equations in the non-orientable case are

$2-g-r=-1$

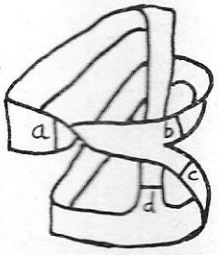
$r=2$

giving $g=1$.

Thus $\mathbb{R}P^2\#2D^2$ (a Möbius band with a hole in it) does the trick.

Solutions to the Miscellaneous Questions

1.



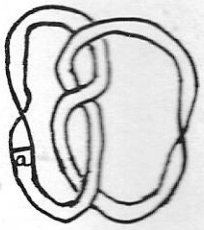
(a) $\chi=8-12+2=-1$

$\beta=1$

The surface is not orientable.

$2\mathbb{R}P^2\#D^2$.

(b)



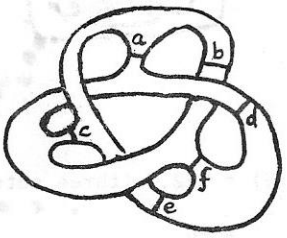
$\chi=2-3+1=0$

$\beta=2$

It is orientable.

It is, in fact, a cylinder!

(c)

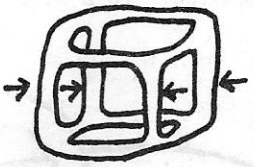


$\chi=12-18+3=-3$

$\beta=3$

It is orientable (a torus with three discs removed).

2. (a) Let S be the given surface, and let S' be obtained by "cutting handles" at each of the four indicated places.

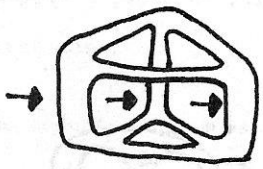


Then S' consists of two disjoint tori and the sum of their Euler characteristics is zero.

$\chi(S)=0-2.4$ (since four cuts were made)

$=-8$.

(b) Again let S be the given surface, and form S' by cutting three handles.



$\chi(S) = 0-2.3$ (three cuts)

$= -6.$

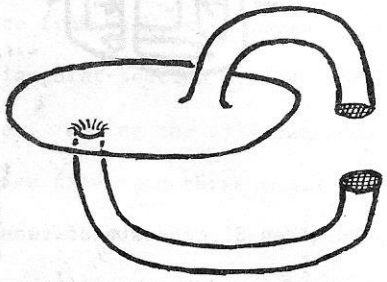
(c) Let S be the given surface, and S' be obtained by cutting 6 handles, three on each layer as in part (b). Then S' is three disjoint tori, and

$\chi(S)=0-2.6=-12.$

(d) Making 3 cuts between each layer gives a total of $3(n-1)$ cuts, so

$\chi(S)=0-2.3(n-1)=6-6n.$

(e) "Cutting the handle" gives a surface homeomorphic to a disc



(shrink the tubes).

Thus $\chi(S)=\chi(D^2)-2$

$=-1.$

Printed in Great Britain by
The Open University